HOLOGRAPHY, GAUGE-GRAVITY CONNECTION AND BLACK HOLE ENTROPY

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Abstract

The issues of holography and possible links with gauge theories in spacetime physics is discussed, in an approach quite distinct from the more restricted AdS-CFT correspondence. A particular notion of holography in the context of black hole thermodynamics is derived (rather than conjectured) from rather elementary considerations, which also leads to a criterion of thermal stability of radiant black holes, without resorting to specific classical metrics. For black holes that obey this criterion, the canonical entropy is expressed in terms of the microcanonical entropy of an Isolated Horizon which is essentially a local generalization of the very global event horizon and is a null inner boundary of spacetime, with marginal outer trapping. It is argued why degrees of freedom on this horizon must be described by a topological gauge theory. Quantizing this boundary theory leads to the microcanonical entropy of the horizon expressed in terms of an infinite series asymptotic in the cross-sectional area, with the leading 'area-law' term followed by finite, unambiguously calculable corrections arising from quantum spacetime fluctuations.

1 Introduction

A black hole spacetime is characterized, as shown in Eddington-Finkelstein coordinates for the Schwarzschild spacetime in Fig. 1. Contracting ellipses depicting a gravitationally collapsing spherical star, are shown to form a marginally outer trapped null surface called the Event Horizon. Local null cones are shown to align with the EH, exhibiting the trapping behaviour characteristic of such spacetimes. These tilt further inside the EH with a shrinkage showing how the causal structure of spacetime is about to disappear at the singularity shown as the dotted vertical line in the middle of the diagram, which is the inevitable fate of the collapsing star and anything else entering the EH.

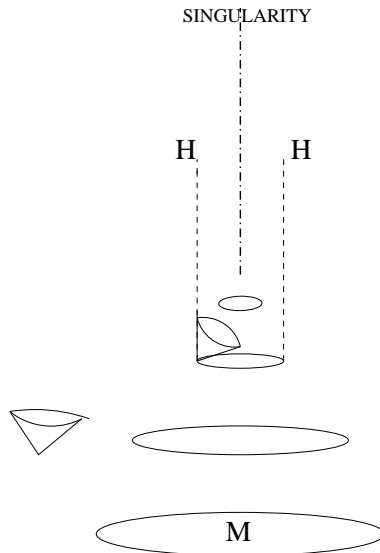**E-mail: parthasarathi.majumdar@saha.ac.in

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Figure 1: Schwarzschild spacetime in Eddington coordinates

An alternative view of the same spacetime is shown in Fig. 2 in conformal coordinates which allows us to discuss asymptopia as a surface in a well-defined part of spacetime. $\mathcal{I}^{\pm}$ are asymptotic future and past null infinities for an asymptotically flat spacetime. A spherical collapse is shown, together with the EH and also the singularity which clearly shows the incompleteness of this spacetime vis-a-vis null and timelike geodesics entering the EH. As shown in the figure, the black hole spacetime is defined as the set of events of the universe $\mathcal{M}$ which do not lie in the chronological past of $\mathcal{I}^+$, i.e., from which information (as light signals say) never make it to future null infinity. This is the region $\mathcal{B}$ shown in the figure. The EH is just the boundary of this spacetime region.

As Subrahmanian Chandrasekhar puts it so eloquently in his treatise Mathematical Theory of Black Holes,
Black holes ... are the most perfect macroscopic objects there are in the universe. The only elements in their construction are our notions of space and time ... and because they appear as ... family of exact solutions of Einstein's equation, they are the simplest objects as well.

Yet black hole spacetimes have

- Singularities, where all known laws of physics break down.

- Event horizon : boundary of spacetime accessible to asymptotic observers.

It is unlikely that black holes can be understood on the basis of **classical GR** *even though their horizons may have macroscopic cross-sectional areas !*

Black holes have a further conundrum associated with the EH: the theorems on Black Hole Mechanics [1] derived from general relativity state that
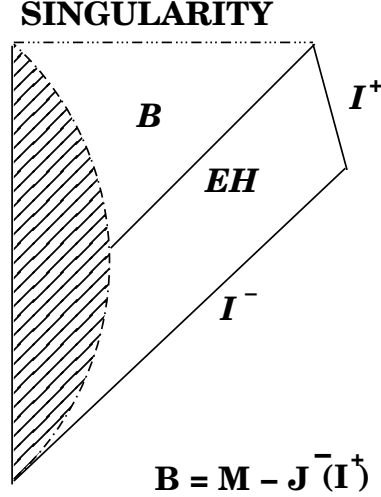
$$\delta \mathcal{A}_{EH} \quad \geq \quad 0$$

SINGULARITY

B

I^+

EH

I^-

$\mathbf{B = M - J^-(I^+)}$

Figure 2: Schwarzschild spacetime: conformal diagram

$$\begin{aligned} \kappa_{EH} &= const \\ \delta M &= \kappa_{EH} \; \delta\mathcal{A}_{EH} + \cdots \end{aligned}$$

While these theorems exhibit an intriguing analogy with the laws of thermodynamics, in reality there is no room for microstates in classical general relativity for a family of exact solutions of Einstein's equation.

In the early 1970s Bekenstein [2] declared that **Black holes (must) have entropy**. The main argument is based on the Generalized Second Law of Thermodynamics : $\delta(S_{out} + S_{bh}) \geq 0$, where S_{out} is the entropy of all matter and radiaton outside the EH. Clearly, the existence of S_{bh} is essential for this law. In order for this entropy to respect the second law of black hole mechanics, independent of black hole parameters, it must be proportional to the horizon area

$$S_{bh} = \frac{\mathcal{A}_{hor}}{4l_P^2} \; (k_B = 1) \tag{1}$$

where, $l_P \equiv (G\hbar/c^3)^{1/2} \sim 10^{-33} cm \Rightarrow$ quantum gravity is necessary to provide the micro states whose counting may eventually lead to black hole entropy. In any case, this implies that one needs to go beyond classical general relativity, in order to make sense of entropy of black hole space times. Thus, black hole physics is by far the more compelling reason for quantizing spacetime geometry than aesthetic reasons based on unification of fundamental interactions, of which there is hardly any evidence observationally.

Two issues that will have to be addressed imperatively before this idea can be implemented:

- **What degrees of freedom contribute to S_{bh} ?**

- **How is it that $S_{bh} = S_{bh}(A_{hor})$ while $S_{thermo} = S_{thermo}(vol)$?**

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2 Holography: a different approach

A possible answer to the second question is provided by the so-called **Holographic Hypothesis** [3], [4], stated as follows [3],

... Given any closed surface, we can represent all that happens (gravitationally) inside it by degrees of freedom on this surface itself. This ... suggests that quantum gravity should be described by a **topological** *quantum field theory in which all (gravitational) degrees of freedom are projected onto the boundary.* However, rather than use this as a working hypothesis, we adopt an alternative viewpoint. We

- Propose : **Holography is an outcome of the diffeomorphism invariance of general relativity**. A version can be *derived* (heuristically).

- We show: how gravitational degrees of freedom are projected to the boundary for a particular model of the boundary known as a Isolated Horizon. We also argue how these boundary degrees of freedom are described by a three dimensional topological gauge theory on the boundary, thus providing an explicit demonstration of a gravitation theory and gauge theory connection. Once again, this is not a conjecture.

- Finally, we discuss important implications of this connection for S_{bh}.

2.1 The proposal

Diffeomorphism invariance $\Rightarrow$ **there are no covariantly conserved energy-momentum tensor for vacuum spacetimes in bulk in full nonlinear general relativity**. Indeed, on the phase space of general relativity, diffeomorphism generators appear as first class constraints. The Hamiltonian for bulk spacetime is expressed as a linear combination of first class constraints,

$$H_v = \int_{\mathcal{S}} [N\mathcal{H} + \mathbf{N} \cdot \mathbf{P}] \tag{2}$$

$$\approx 0 \ when \ \mathcal{H} \approx 0, \ \mathbf{P} \approx 0 \tag{3}$$

where $\mathcal{H}, \mathbf{P}$ are diffeomorphism generators and $N(\text{lapse}), \mathbf{N}(\text{shift})$ are Lagrange multipliers. In other words, there is no analogue of $\mathbf{E}^2 + \mathbf{B}^2$ in vacuum general relativity in the bulk. Thus,

$$H_{GR} = \underbrace{H_v}_{bulk} + \underbrace{H_b}_{boundary} . \tag{4}$$

On the constraint surface, $H_{GR} \approx H_b$, which implies that primary excitations of quantum general relativity are not particle-like, but extended, like non-perturbative quantum chromodynamics.

But, what about gravitons ? They are of course particle excitations of perturbative quantum gravity, around weak gravitational backgrounds :

$$g_{ab} = \underbrace{\hat{g}_{ab}}_{fixed \ bkgd} + \underbrace{h_{ab}}_{graviton} \tag{5}$$

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Thus, the description of gravitons requires

- a *fixed nondynamical* background

- an expansion around a fiducial background, which is sensible only perturbatively

- as such, it is quite inadequate for black hole thermodynamics.

In other words, black hole thermodynamics is *not* the thermodynamics of a gas of gravitons in a non-dynamical gravitational background. It is rather the thermodynamics of a black hole spacetime itself, i.e., of the geometry; this is only possible if one can ascribe quantum states to spacetime geometry which can be counted as microstates.

2.2 'Thermal' holography

We now consider a canonical ensemble of radiant black hole spacetimes in contact with a radiation bath at an inverse temperature β. The canonical partition function is given by

$$Z(\beta) \;=\; Tr \exp -\beta \hat{H} \;, \tag{6}$$

where,

$$\hat{H} \;=\; \underbrace{\hat{H}_v}_{blk} \;+\; \underbrace{\hat{H}_b}_{bdy} \;. \tag{7}$$

The Tr is over states defined as

$$|\Psi\rangle \;=\; \sum_{v,b} c_{vb} \underbrace{|\psi_v\rangle}_{blk} \underbrace{|\chi_b\rangle}_{bdy} \tag{8}$$

i.e., the full Hilbert space $\mathcal{H} = \mathcal{H}_v \otimes \mathcal{H}_b$. The **Hamiltonian constraint** in the bulk implies that the quantum Hamiltonian operator annihilates the bulk quantum states

$$\hat{H}_v \, |\psi_v\rangle \;=\; 0 \;. \tag{9}$$

It follows that

$$\begin{aligned}
Z(\beta) \;&=\; \sum_b \left(\sum_v |c_{vb}|^2 || \, |\psi_v\rangle \, ||^2 \right) \langle \chi_b | \exp -\beta \hat{H}_{bdy} | \chi_b \rangle \\
&=\; Tr_{bdy} \exp -\beta \hat{H}_{bdy} \\
&\equiv\; Z_{bdy} \;. \tag{10}
\end{aligned}$$

In other words, **the bulk states decouple!** Boundary states determine bh thermodynamics completely : a thermal version of **holography !** This is different from the holographic hypothesis quoted above wherein *all* bulk states are stipulated to be projected onto the boundary.
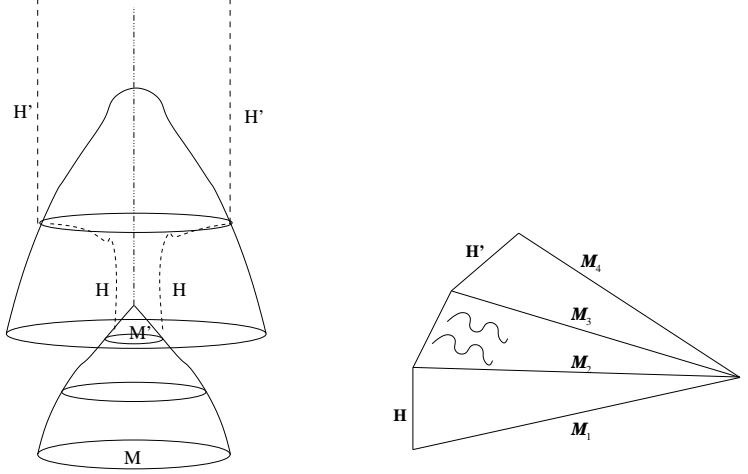
Figure 3: Isolated Horizons

3   Isolated Horizons

So far no specification of the kind of spacetime boundary we have in mind has been made. Clearly, our interest is not in the asymptotic boundary. Instead we focus on an *inner* boundary of spacetime. Recall that the event horizon itself is a boundary of the chronological past of future asymptopia. But the event horizon is too global for our purpose. It has the following lacunae:

- EH is *teleological* in nature, i.e., it is determined only after *entire* spacetime is known.

- Stationarity $\Rightarrow$ black hole metric has a *global* timelike isometry.

- Cosmological horizons (like the de Sitter horizon) cannot be characterized as event horizons.

- The (ADM) mass of the black hole is not defined on the event horizon but as an integral over spatial infinity (i^0 in Fig. 2).

In view of these shortcomings, we seek a *local* generalization of event horizons.

Such an alternative has already been found [5] and is called an Isolated Horizon (IQ).

We summarize the main properties of such a horizon, referring the reader to [5] for more details.

- An IQ has no global timelike isometry $\Rightarrow$ it is a **nonstationary** generalization of stationary event horizons, cosmological horizons, etc., allowing radiation to exist infitesimally close to it.

- It is a null inner boundary of spacetime with topol $\mathbf{R} \otimes \mathbf{S}^2$.

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- The cross-sectional area $\mathcal{A}_{IH}$ of an IH remains constant: this is precisely the *isolation*. Thus, nothing ever crosses an IH.

- *Zeroth law of IH Mechanics* The surface gravity $\kappa_{IH} = const$

- On IH, can define mass $M_{IH} = M_{ADM} - \mathcal{E}_{rad}^{\infty}$ such that $\delta M_{IH} = \kappa_l \delta A_{hor} + \dots$ *(Ist law of IHM)*

- Such horizons correspond thermodynamically to a *microcanonical ensemble with fixed $\mathcal{A}_{IH}$* .

3.1 Canonical entropy and thermal stability

Consider now a canonical ensemble of IHs in contact with radiation; we proceed to compute S_{can} for this ensemble, assuming the equilibrium configuration to be an IH with fixed A_{IH} and $M_{IH} = M(A_{IH})$. Retaining Gaussian fluctuations around a saddle point chosen to be this equilibrium configuration, we get [6], [7]

$$S_{can} = S_{IH} + \frac{1}{2} \log S_{IH} \qquad (11)$$

where S_{IH} is the *microcanonical* entropy of the equilibrium IH. Two issues arise immediately:

- What is S_{IH} ?

- $S_{can} > 0 \Rightarrow$ black hole is thermally stable, i.e., heat capacity $C > 0$. Under what conditions does this happen ?

We answer the second question first: the condition for thermal stability has been determined [8] [9]

$$\frac{M_{IH}}{M_P} > \frac{S_{IH}}{k_B} \qquad (12)$$

This turns out to be the **necessary and sufficient cond. for $S_{can} > 0$ and $C > 0$.** Saturation of the inequality is seen to lead to $C \nearrow \infty$! This is reminiscent of a first order phase transition, even though here the transition is between *a stable and an unstable phase*. This is similar to the Hawking-Page transition [10] for an AdS-Schwarzschild black hole. The important distinction here is that it is completely general and also to an extent quantum in nature, in contradistinction to the Hawking-Page treatment which is restricted to a semiclassical analysis in anti-de Sitter black hole spacetime. It is seemingly generalizable to more general black holes with charge and angular momentum, within the grand canonical ensemble.

3.2 IH as a null boundary: gravity-gauge link

Because IH is an *inner* boundary, we must add boundary term to the action in order that the variational principle canbe used to derive Einstein's equation. Thus,

$$\mathcal{S} = \mathcal{S}_{EHL} + \mathcal{S}_{IH} \tag{13}$$

such that

$$\delta\mathcal{S}_{EHL}|_{IH} + \delta\mathcal{S}_{IH} = 0 \tag{14}$$

Since the IH is null the induced metric on it is degenerate $\sqrt{^3 g_{IH}} = 0$. This has the consequence that the quantum theory describing IH degrees of freedom must be a three dimensional topological field theory for which the action is indep. of $^3 g_{IH}$. Which 3 dim topological field theory ? It must be a theory such that the degrees of freedom are related in some manner to the bulk spacetime degrees of freedom (metric, tetrad, connection).

It turns out that with GR formulated in bulk as a gauge theory of the Poincare group (and diffeomorphisms), the theory induced by the boundary conditions IH is an $SU(2)$ Chern Simons gauge theory (in *time* gauge where local Lorentz boosts are gauge fixed) with coupling constant $k \equiv A_{IH}/4\pi l_P^2 >>> 1$ [11]. This provides one of the clearest examples of a gravity-gauge theory link in the literature. This connection is based on far firmer footing that others based on conjectured relationships. Using $\mathcal{S} = \mathcal{S}_{EHL} + S_{IH}$, the variational principle works, provided the following *consistency* condition holds

$$\left(\frac{k}{2\pi} F_{CS} + E \times E \right)_{S^2} = 0 \ . \tag{15}$$

This is nothing but the Chern Simons theory equation of motion with the second term functioning as source currents. This implies that the bulk spatial geometry characterized by E plays the role of source for the Chern Simons degrees of freedom (given by F_{CS}) characterizing boundary (IH) geometry. It is also a precise demonstration of the *projection of bulk gravitational degrees of freedom to the boundary* hypothesized in the Holographic Hypothesis.

4 Microcanonical entropy

4.1 Loop Quantum Gravity : spin network basis

We now address the question of the microcanonical entropy of the IH (S_{IH}). The calculation follows the approach and methodology laid out in [11] - [14]. It is based on Loop Quantum Gravity as well as the connection between Chern Simons gauge theories and Wess-Zumino-Witten models [15]. Loop Quantum Gravity is perhaps the only known quantum theory of spacetime geometry, which is background-independent and non-perturbative. It is a canonical version of quantum general relativity, describing quantum three dimensional space (on every spatial slice) in terms of *Spin Network* states. The
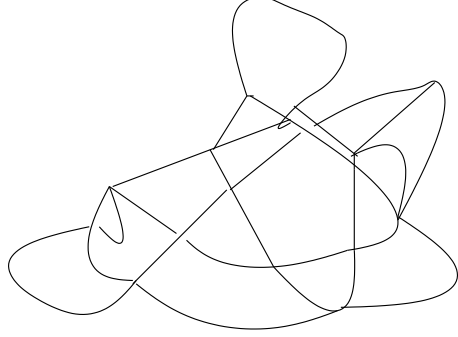
Figure 4: Spin network graph

Spin Network basis was first proposed by Penrose and adapted to loop quantum gravity by Rovelli and Smolin [16]. Three dimensional space is supposed to consist of fluctuating network graphs whose links each carry an $SU(2)$ irreducible representation index ('spin', $j = 0, 1/2, 1, 3/2, \ldots$). Links meet at vertices containing invariant $SU(2)$ 'transporter' tensors constructed out of the Levi-Civita tensor, depending on the valence of each vertex. An arbitrary quantum state is a superposition of spin network states which form an overcomplete basis.

The great advantage of the spin network basis is that geometric observables (represented as self-adjoint operators), like length, area, volume, are diagonal in this basis and turn out to have *discrete* spectra. In particular, consider a spacelike two surface inserted into an arbitrary spin network graph. The actual area of this surface will fluctuate around the classical area A_{cl} by terms $O(l_P^2)$ when the graph fluctuates, with different spins puncturing the two-surface and transfering their spins to the punctures.

The area operator, defined as

$$\hat{A}_S \equiv \sum_{I=1}^{N} \int_{S_I} \det^{1/2}[^2g(\hat{E})] \ /labelarop \tag{16}$$

can be shown [16] to possess the bounded, discrete spectrum

$$a(j_1, \ldots, j_N) \quad = \quad \frac{1}{4}\gamma l_P^2 \sum_{p=1}^{N} \sqrt{j_p(j_p + 1)} \tag{17}$$

$$\lim_{N \to \infty} a(j_1, \ldots. j_N) \quad \leq \quad A_{cl} + O(l_P^2) \ . \tag{18}$$

4.2   'Quantum' Isolated Horizon

Loop quantum gravity has not yet reached a stage of development where one can unambiguously exhibit an IH formation from an appropriate solution of the quantum Einstein (Wheeler-de Witt) equation, in some semiclassical approximation. Instead, we adopt an *effective theory* viewpoint whereby we insert a foliation of the IH into the spin network
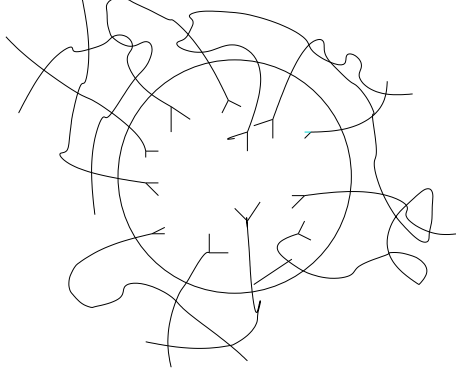
Figure 5: Quantum isolated horizon

characterizing quantum spatial geometry, and use the formalism of Chern Simons theory to obtain the states on this spherical section of the IH, with point sources carrying spin j (arbitrary) on the punctures. Our interest is to count $\dim \mathcal{H}_{CS+ptsources(j_1,...j_n)}$ and get S_{IH} from

$$S_{IH} \equiv \log \dim \mathcal{H}_{CS+(j_1,...,j_n)} \ , \tag{19}$$

for some fixed $A_{IH} >> l_P^2$, *restricting to only states with vanishing total spin.* This latter restriction is enforced by the $SU(2)$ Gauss law constraint which implies that only rotationally invariant states are physical.

This computation is simplified by the relation [15] between the dimensionality of the CS theory Hilbert space and *the conformal blocks of an $SU(2)_k$ WZW model living on the punctured 2-sphere.* Using this relation, and also the Verlinde formula, the dimensionality of the Chern Simons Hilbert space is given by [12]

$$\dim \mathcal{H}_{CS+(j_1,...,j_n)} = \prod_{p=1}^{n} \sum_{m_p=-j_p}^{j_p} [\delta_{m_1+\cdots+m_n,0} - \frac{1}{2}\delta_{m_1+\cdots+m_n,-1} \tag{20}$$

$$- \quad \frac{1}{2}\delta_{m_1+\cdots+m_n,1}] \ . \tag{21}$$

A moment's reflection on eq. (21) is adequate to persuade us that indeed the states with vanishing composite spin must have not only $m = 0$ but discounted by those states which have integral composite spin; the latter have not only an $m = 0$ sector, but also $m = \pm 1$ sectors. These nonvanishing composite spin states do not satisfy the Gauss law constraint and have to be eliminated if we are to consider only spinless states as physical. Without this elimination, we have a larger degeneracy which will ensue if the residual gauge invariance is $U(1)$ [11] rather than $SU(2)$. The reason we think it is natural to take $SU(2)$ rather than $U(1)$ [11] as the remnant of the local Lorentz invariance is that the former is the invariance group relevant to the Gauss law constraint on the entire spacetime, once Lorentz boosts are frozen out by choosing the 'time' gauge. A further

gauge fixing to $U(1)$ on the IH [11] appears to us to be overly restrictive formally. Of course, one may desire to obtain the degeneracy of the Chern Simons states for the entire Lorentz group as the gauge group on the IH, but that task is made difficult by the fact that unitary irreps of the Lorentz group are infinite dimensional.

If, for simplicity we choose $j_p = \frac{1}{2} \; \forall \; p = 1, \ldots, n$ we get

$$S_{mc} = S_{IH} = \underbrace{\frac{A_{IH}}{4l_P^2}}_{\text{Ashtekar et.al. 97}} \qquad (22)$$

$$\underbrace{- \; \frac{3}{2} \log \left(\frac{A_{IH}}{4l_P^2} \right) \; + \; const. \; + \; O(A_{IH}^{-1})}_{\text{Kaul \& PM 2000}} \; . \qquad (23)$$

The remarkable aspect of (23) is that, perhaps for the first time since Bekenstein's pioneering work, one has an ab initio computation of S_{IH} and obtained an infinite series, asymptotic in A_{IH}, of quantum spacetime fluctuation corrections to the Bekenstein-Hawking area law; each term of this series is finite and unambiguously calculable. The leading correction to the area law is logarithmic and has what appears to be a robust coefficient. With due modesty, one may say that these corrections are the only known *physical* signatures of loop quantum gravity as applied to the computation of *microcanonical* black hole entropy.

5 Pending Issues

- One needs to go beyond effective description in terms of an embedded IH : we need to solve quantum dynamics and show the formation of the horizon.

- We need to determine if Hawking radiation from IH is at all possible, given its isolation.

- We need to determine if the thermal nature of Hawking radiation spectrum is an artifact of the semiclassical approximation inherent in the pioneering work. In other words, if a version of the horizon is shown to radiate, then within a full quantum description, is the radiation of quanta still in a thermal distribution ?

- We need to understand if the lowest area quantum $\sim l_P^2$ has implications for the *information loss problem.*

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